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# CONCENTRATING STANDING WAVES FOR THE FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION

JUAN DÁVILA, MANUEL DEL PINO, AND JUNCHENG WEI

ABSTRACT. We consider the semilinear equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N)$$

where  $0 < s < 1$ ,  $1 < p < \frac{N+2s}{N-2s}$ ,  $V(x)$  is a sufficiently smooth potential with  $\inf_{\mathbb{R}} V(x) > 0$ , and  $\varepsilon > 0$  is a small number. Letting  $w_\lambda$  be the radial ground state of  $(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0$  in  $H^{2s}(\mathbb{R}^N)$ , we build solutions of the form

$$u_\varepsilon(x) \sim \sum_{i=1}^k w_{\lambda_i}((x - \xi_i^\varepsilon)/\varepsilon),$$

where  $\lambda_i = V(\xi_i^\varepsilon)$  and the  $\xi_i^\varepsilon$  approach suitable critical points of  $V$ . Via a Lyapunov Schmidt variational reduction, we recover various existence results already known for the case  $s = 1$ . In particular such a solution exists around  $k$  nondegenerate critical points of  $V$ . For  $s = 1$  this corresponds to the classical results by Floer-Weinstein [13] and Oh [24, 25].

## 1. INTRODUCTION AND MAIN RESULTS

We consider the fractional nonlinear Schrödinger equation

$$i\hbar\psi_t = \hbar^{2s}(-\Delta)^s \psi + W(x)\psi - |\psi|^{p-1}\psi \quad (1.1)$$

where  $(-\Delta)^s$ ,  $0 < s < 1$ , denotes the usual fractional Laplace operator,  $W(x)$  is a bounded potential,  $p > 1$  and  $\hbar$  designates the usual Planck constant. Equation (1.1) was introduced by Laskin [19] as an extension of the the classical nonlinear Schrödinger equation  $s = 1$  in which the Brownian motion of the quantum paths is replaced by a Lévy flight. Here  $\psi = \psi(x, t)$  represents the quantum mechanical probability amplitude for a given unit-mass particle to have position  $x$  at time  $t$  (the corresponding probability density is  $|\psi|^2$ ), under a confinement due to the potential  $W$ . We refer to [19, 20, 21] for detailed physical discussions and motivation of equation (1.1).

We are interested in the *semi-classical limit* regime,  $0 < \varepsilon := \hbar \ll 1$ . For small values of  $\varepsilon$  the wave function tends to concentrate as a material particle.

Our purpose is to find *standing-wave solutions* of (1.1), which are those of the form  $\psi(x, t) = u(x)e^{iEt/\varepsilon}$  with  $u(x)$  a real-valued function. Letting  $V(x) = W(x) + E$ , equation (1.1) becomes

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N. \quad (1.2)$$

We assume in what follows that  $V$  satisfies

$$V \in C^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \inf_{\mathbb{R}^N} V(x) > 0. \quad (1.3)$$

We are interested in finding solutions with a *spike pattern* concentrating around a finite number of points in space as  $\varepsilon \rightarrow 0$ . This has been the topic of many works in the standard case  $s = 1$ , relating the concentration points with critical points of the potential, starting in 1986 with the pioneering work by Floer and Weinstein [13], then continued by Oh [24, 25]. The natural place to look for solutions to (1.2) that decay at infinity is the space  $H^{2s}(\mathbb{R}^N)$ , of all functions  $u \in L^2(\mathbb{R}^N)$  such that

$$\int_{\mathbb{R}^N} (1 + |\xi|^{4s}) |\hat{u}(\xi)|^2 d\xi < +\infty,$$

where  $\hat{\cdot}$  denotes Fourier transform. The fractional Laplacian  $(-\Delta)^s u$  of a function  $u \in H^{2s}(\mathbb{R}^N)$  is defined in terms of its Fourier transform by the relation

$$\widehat{(-\Delta)^s u} = |\xi|^{2s} \hat{u} \in L^2(\mathbb{R}^N).$$

We will explain next what we mean by a *spike pattern* solution of equation (1.2). Let us consider the basic problem

$$(-\Delta)^s v + v - |v|^{p-1} v = 0, \quad v \in H^{2s}(\mathbb{R}^N). \quad (1.4)$$

We assume the following constraint in  $p$ ,

$$1 < p < \begin{cases} \frac{N+2s}{N-2s} & \text{if } 2s < N, \\ +\infty & \text{if } 2s \geq N. \end{cases} \quad (1.5)$$

Under this condition it is known the existence of a positive, radial *least energy solution*  $v = w(x)$ , which gives the lowest possible value for the energy

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^N} v(-\Delta)^s v + \frac{1}{2} \int_{\mathbb{R}^N} v^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |v|^{p+1}.$$

among all nontrivial solutions of (1.4). An important property, which has only been proven recently by Frank-Lenzman-Silvestre [15] (see also [2, 14]), is that there exists a radial least energy solution which is nondegenerate, in the sense that the space of solutions of the equation

$$(-\Delta)^s \phi + \phi - pw^{p-1}\phi = 0, \quad \phi \in H^{2s}(\mathbb{R}^n) \quad (1.6)$$

consists of exactly of the linear combinations of the translation-generators,  $\frac{\partial w}{\partial x_j}$ ,  $j = 1, \dots, N$ .

It is easy to see that the function

$$w_\lambda(x) := \lambda^{\frac{1}{p-1}} w(\lambda^{\frac{1}{2s}} x)$$

satisfies the equation

$$(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0 \quad \text{in } \mathbb{R}^N.$$

Therefore for any point  $\xi \in \mathbb{R}^N$ , taking  $\lambda = V(\xi)$ , the spike-shape function

$$u(x) = w_{V(\xi)} \left( \frac{x - \xi}{\varepsilon} \right) \quad (1.7)$$

satisfies

$$\varepsilon^{2s} (-\Delta)^s u + V(\xi)u - u^p = 0.$$

Since the  $\varepsilon$ -scaling makes it *concentrate* around  $\xi$ , this function constitutes a good positive approximate solution to equation (1.2), namely of

$$\varepsilon^{2s} (-\Delta)^s u + V(x)u - u^p = 0, \quad (1.8)$$

$$u > 0, \quad u \in H^{2s}(\mathbb{R}^N).$$

We call a *k-spike pattern solution* of (1.8) one that looks approximately like a superposition of *k* spikes like (1.7), namely a solution  $u_\varepsilon$  of the form

$$u_\varepsilon(x) = \sum_{i=1}^k w_{V(\xi_i^\varepsilon)} \left( \frac{x - \xi_i^\varepsilon}{\varepsilon} \right) + o(1) \quad (1.9)$$

for points  $\xi_1^\varepsilon, \dots, \xi_k^\varepsilon$ , where  $o(1) \rightarrow 0$  in  $H^{2s}(\mathbb{R}^N)$  as  $\varepsilon \rightarrow 0$ .

In what follows we assume that  $p$  satisfies condition (1.5) and  $V$  condition (1.3).

Our first result concerns the existence of multiple spike solution at separate places in the case of stable critical points.

**Theorem 1.** *Let  $\Lambda_i \subset \mathbb{R}^N$ ,  $i = 1, \dots, k$ ,  $k \geq 1$  be disjoint bounded open sets in  $\mathbb{R}^N$ . Assume that*

$$\deg(\nabla V, \Lambda_i, 0) \neq 0 \quad \text{for all } i = 1, \dots, k.$$

*Then for all sufficiently small  $\varepsilon$ , Problem (1.8) has a solution of the form (1.9) where  $\xi_i^\varepsilon \in \Lambda_i$  and*

$$\nabla V(\xi_i^\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

An immediate consequence of Theorem 1 is the following.

**Corollary 1.1.** *Assume that  $V$  is of class  $C^2$ . Let  $\xi_1^0, \dots, \xi_k^0$  be  $k$  non-degenerate critical points of  $V$ , namely*

$$\nabla V(\xi_i^0) = 0, \quad D^2V(\xi_i^0) \text{ is invertible for all } i = 1, \dots, k.$$

*Then, a  $k$ -spike solution of (1.8) of the form (1.9) with  $\xi_i^\varepsilon \rightarrow \xi_i^0$  exists.*

When  $s = 1$ , the result of Corollary 1.1 is due to Floer and Weinstein [13] for  $N = 1$  and  $k = 1$  and to Oh [24, 25] when  $N \geq 1$ ,  $k \geq 1$ . Theorem 1 for  $s = 1$  was proven by Yanyan Li [22].

**Remark 1.1.** As the proof will yield, Theorem 1 for  $0 < s < 1$  holds true under the following, more general condition introduced in [22]. Let  $\Lambda = \Lambda_1 \times \dots \times \Lambda_k$  and assume that the function

$$\varphi(\xi_1, \dots, \xi_k) = \sum_{i=1}^k V(\xi_i)^\theta, \quad \theta = \frac{p+1}{p-1} - \frac{N}{2s} > 0 \quad (1.10)$$

has a *stable critical point situation* in  $\Lambda$ : there is a number  $\delta_0 > 0$  such that for each  $g \in C^1(\bar{\Omega})$  with  $\|g\|_{L^\infty(\Lambda)} + \|\nabla g\|_{L^\infty(\Lambda)} < \delta_0$ , there is a  $\xi_g \in \Lambda$  such that  $\nabla \varphi(\xi_g) + \nabla g(\xi_g) = 0$ . Then for all sufficiently small  $\varepsilon$ , Problem (1.8) has a solution of the form (1.9) where  $\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_k^\varepsilon) \in \Lambda$  and  $\nabla \varphi(\xi^\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

**Theorem 2.** *Let  $\Lambda$  be a bounded, open set with smooth boundary such that  $V$  is such that either*

$$c = \inf_{\Lambda} V < \inf_{\partial\Lambda} V \quad (1.11)$$

*or*

$$c = \sup_{\Lambda} V > \sup_{\partial\Lambda} V$$

or, there exist closed sets  $B_0 \subset B \subset \Lambda$  such that

$$c = \inf_{\Phi \in \Gamma} \sup_{x \in B} V(\Phi(x)) > \sup_{B_0} V. \quad (1.12)$$

where  $\Gamma = \{\Phi \in C(B, \bar{\Lambda}) \mid \Phi|_{B_0} = Id\}$  and  $\nabla V(x) \cdot \tau \neq 0$  for all  $x \in \partial\Lambda$  with  $V(x) = c$  and some tangent vector  $\tau$  to  $\partial\Lambda$  at  $x$ .

Then, there exists a 1-spike solution of (1.8) with  $\xi^\varepsilon \in \Lambda$  with  $\nabla V(\xi^\varepsilon) \rightarrow 0$  and  $V(\xi^\varepsilon) \rightarrow c$ .

In the case  $s = 1$ , the above results were found by del Pino and Felmer [7, 8]. The case of a (possibly degenerate) global minimizer was previously considered by Rabinowitz [26] and X. Wang [28]. An isolated maximum with a power type degeneracy appears in Ambrosetti, Badiale and Cingolani [1]. Condition (1.12) is called a *nontrivial linking situation* for  $V$ . The cases of  $k$  disjoint sets where (1.11) holds was treated in [9, 17]. Multiple spikes for disjoint nontrivial linking regions were first considered in [10], see also [5, 16] for other multiplicity results.

Our last result concerns the existence of multiple spikes *at the same point*.

**Theorem 3.** *Let  $\Lambda$  be a bounded, open set with smooth boundary such that  $V$  is such that*

$$\sup_{\Lambda} V > \sup_{\partial\Lambda} V.$$

*Then for any positive integer  $k$  there exists a  $k$ -spike solution of (1.8) with spikes  $\xi_j^\varepsilon \in \Lambda$  satisfying  $V(\xi_j^\varepsilon) \rightarrow \max_{\Lambda} V$ .*

In the case  $s = 1$ , Theorem 3 was proved by Kang and Wei [18]. D'Aprile and Ruiz [6] have found a phenomenon of this type at a saddle point of  $V$ .

The rest of this paper will be devoted to the proofs of Theorems 1–3. The method of construction of a  $k$ -spike solution consists of a Lyapunov-Schmidt reduction in which the full problem is reduced to that of finding a critical point  $\xi^\varepsilon$  of a functional which is a small  $C^1$ -perturbation of  $\varphi$  in (1.10). In this reduction the nondegeneracy result in [15] is a key ingredient.

After this has been done, the results follow directly from standard degree theoretical or variational arguments. The Lyapunov-Schmidt reduction is a method widely used in elliptic singular perturbation problems. Some results of variational type for  $0 < s < 1$  have been obtained for instance in [12] and [27]. We believe that the scheme of this paper may be generalized to concentration on higher dimensional regions, while that could be much more challenging. See [11, 23] for concentration along a curve in the plane and  $s = 1$ .

## 2. GENERALITIES

Let  $0 < s < 1$ . Various definitions of the fractional Laplacian  $(-\Delta)^s \phi$  of a function  $\phi$  defined in  $\mathbb{R}^N$  are available, depending on its regularity and growth properties.

As we have recalled in the introduction, for  $\phi \in H^{2s}(\mathbb{R}^N)$  the standard definition is given via Fourier transform  $\widehat{\cdot}$ .  $(-\Delta)^s \phi \in L^2(\mathbb{R}^N)$  is defined by the formula

$$|\xi|^{2s} \widehat{\phi}(\xi) = \widehat{(-\Delta)^s \phi}. \quad (2.1)$$

When  $\phi$  is assumed in addition sufficiently regular, we obtain the direct representation

$$(-\Delta)^s \phi(x) = d_{s,N} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x - y|^{N+2s}} dy \quad (2.2)$$

for a suitable constant  $d_{s,N}$  and the integral is understood in a principal value sense. This integral makes sense directly when  $s < \frac{1}{2}$  and  $\phi \in C^{0,\alpha}(\mathbb{R}^N)$  with  $\alpha > 2s$ , or if  $\phi \in C^{1,\alpha}(\mathbb{R}^N)$ ,  $1 + \alpha > 2s$ . In the latter case, we can desingularize the integral representing it in the form

$$(-\Delta)^s \phi(x) = d_{s,N} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y) - \nabla \phi(x)(x - y)}{|x - y|^{N+2s}} dy.$$

Another useful (local) representation, found by Caffarelli and Silvestre [3], is via the following boundary value problem in the half space  $\mathbb{R}_+^{N+1} = \{(x, y) / x \in \mathbb{R}^N, y > 0\}$ :

$$\begin{cases} \nabla \cdot (y^{1-2s} \nabla \tilde{\phi}) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \tilde{\phi}(x, 0) = \phi(x) & \text{on } \mathbb{R}^N. \end{cases}$$

Here  $\tilde{\phi}$  is the  $s$ -harmonic extension of  $\phi$ , explicitly given as a convolution integral with the  $s$ -Poisson kernel  $p_s(x, y)$ ,

$$\tilde{\phi}(x, y) = \int_{\mathbb{R}^N} p_s(x - z, y) \phi(z) dz,$$

where

$$p_s(x, y) = c_{N,s} \frac{y^{4s-1}}{(|x|^2 + |y|^2)^{\frac{N-1+4s}{2}}}$$

and  $c_{N,s}$  achieves  $\int_{\mathbb{R}^N} p(x, y) dx = 1$ . Then under suitable regularity,  $(-\Delta)^s \phi$  is the Dirichlet-to-Neumann map for this problem, namely

$$(-\Delta)^s \phi(x) = \lim_{y \rightarrow 0^+} y^{1-2s} \partial_y \tilde{\phi}(x, y). \quad (2.3)$$

Characterizations (2.1), (2.2), (2.3) are all equivalent for instance in Schwartz's space of rapidly decreasing smooth functions.

Let us consider now for a number  $m > 0$  and  $g \in L^2(\mathbb{R}^N)$  the equation

$$(-\Delta)^s \phi + m\phi = g \quad \text{in } \mathbb{R}^N.$$

Then in terms of Fourier transform, this problem, for  $\phi \in L^2$ , reads

$$(|\xi|^{2s} + m) \hat{\phi} = \hat{g}$$

and has a unique solution  $\phi \in H^{2s}(\mathbb{R}^N)$  given by the convolution

$$\phi(x) = T_m[g] := \int_{\mathbb{R}^N} k(x - z) g(z) dz, \quad (2.4)$$

where

$$\hat{k}(\xi) = \frac{1}{|\xi|^{2s} + m}.$$

Using the characterization (2.3) written in weak form,  $\phi$  can then be characterized by  $\phi(x) = \tilde{\phi}(x, 0)$  in trace sense, where  $\tilde{\phi} \in H$  is the unique solution of

$$\iint_{\mathbb{R}_+^{N+1}} \nabla \tilde{\phi} \nabla \varphi y^{1-2s} + m \int_{\mathbb{R}^N} \phi \varphi = \int_{\mathbb{R}^N} g \varphi, \quad \text{for all } \varphi \in H, \quad (2.5)$$

where  $H$  is the Hilbert space of functions  $\varphi \in H_{loc}^1(\mathbb{R}_+^{N+1})$  such that

$$\|\varphi\|_H^2 := \iint_{\mathbb{R}_+^{N+1}} |\nabla \varphi|^2 y^{1-2s} + m \int_{\mathbb{R}^N} |\varphi|^2 < +\infty,$$

or equivalently the closure of the set of all functions in  $C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$  under this norm.

A useful fact for our purposes is the equivalence of the representations (2.4) and (2.5) for  $g \in L^2(\mathbb{R}^N)$ .

**Lemma 2.1.** *Let  $g \in L^2(\mathbb{R}^N)$ . Then the unique solution  $\tilde{\phi} \in H$  of Problem (2.5) is given by the  $s$ -harmonic extension of the function  $\phi = T_m[g] = k * g$ .*

*Proof.* Let us assume first that  $\hat{g} \in C_c^\infty(\mathbb{R}^N)$ . Then  $\phi$  given by (2.4) belongs to  $H^{2s}(\mathbb{R}^N)$ . Take a test function  $\psi \in C_c^\infty(\mathbb{R}_+^{N+1})$ . Then the well-known computation by Caffarelli and Silvestre shows that

$$\begin{aligned} \iint_{\mathbb{R}_+^{N+1}} \nabla \tilde{\phi} \nabla \psi y^{1-2s} dy dx &= \\ \int_{\mathbb{R}^N} \lim_{y \rightarrow 0} y^{1-2s} \partial_y \tilde{\phi}(y, \cdot) \psi dx &= \int_{\mathbb{R}^N} \psi (-\Delta)^s \phi dx = \int_{\mathbb{R}^N} (g - m\phi) dx. \end{aligned}$$

By taking  $\psi = \tilde{\phi} \eta_R$  for a suitable sequence of smooth cut-off functions equal to one on expanding balls  $B_R(0)$  in  $\mathbb{R}_+^{N+1}$ , and using the behavior at infinity of  $\tilde{\phi}$  which resembles the Poisson kernel  $p_s(x, y)$ , we obtain

$$\iint_{\mathbb{R}_+^{N+1}} |\nabla \tilde{\phi}|^2 y^{1-2s} dy dx + m \int_{\mathbb{R}^N} |\phi|^2 = \int_{\mathbb{R}^N} g \phi$$

and hence  $\|\tilde{\phi}\|_H \leq C \|g\|_{L^2}$  and satisfies (2.5). By density, this fact extends to all  $g \in L^2(\mathbb{R}^N)$ . The result follows since the solution of Problem (2.5) in  $H$  is unique.  $\square$

Let us recall the main properties of the fundamental solution  $k(x)$  in the representation (2.4), which are stated for instance in [15] or in [12].

We have that  $k$  is radially symmetric and positive,  $k \in C^\infty(\mathbb{R}^N \setminus \{0\})$  satisfying

•

$$|k(x)| + |x| |\nabla k(x)| \leq \frac{C}{|x|^{N-2s}} \quad \text{for all } |x| \leq 1,$$

•

$$\lim_{|x| \rightarrow \infty} k(x) |x|^{N+2s} = \gamma > 0,$$

•

$$|x| |\nabla k(x)| \leq \frac{C}{|x|^{N+2s}} \quad \text{for all } |x| \geq 1.$$

The operator  $T_m$  is not just defined on functions in  $L^2$ . For instance it acts nicely on bounded functions. The positive kernel  $k$  satisfies  $\int_{\mathbb{R}^N} k = \frac{1}{m}$ . We see that if  $g \in L^\infty(\mathbb{R}^N)$  then

$$\|T_m[g]\|_\infty \leq \frac{1}{m} \|g\|_\infty.$$

We have indeed the validity of an estimate like this for  $L^\infty$  weighted norms as follows.

**Lemma 2.2.** *Let  $0 \leq \mu < N + 2s$ . Then there exists a  $C > 0$  such that*

$$\|(1 + |x|)^\mu T_m[g]\|_{L^\infty(\mathbb{R}^N)} \leq C \|(1 + |x|)^\mu g\|_{L^\infty(\mathbb{R}^N)}.$$

*Proof.* Let us assume that  $0 \leq \mu < N + 2s$  and let  $\bar{g}(x) = \frac{1}{(1+|x|)^\mu}$ . Then

$$T[\bar{g}](x) = \int_{|y-x| < \frac{1}{2}|x|} \frac{k(y)}{(1+|y-x|)^\mu} dy + \int_{|y-x| > \frac{1}{2}|x|} \frac{k(y)}{(1+|y-x|)^\mu} dy.$$

Then, as  $|x| \rightarrow \infty$  we find

$$|x|^\mu \int_{|y-x| < \frac{1}{2}|x|} \frac{k(y)}{(1+|y-x|)^\mu} dy \sim |x|^{-2s} \rightarrow 0,$$

and since  $k \in L^1(\mathbb{R}^N)$ , by dominated convergence we find that as  $|x| \rightarrow \infty$

$$\int_{|x-y| > \frac{1}{2}|x|} \frac{k(y)|x|^\mu}{(1+|x-y|)^\mu} dy \rightarrow \int_{\mathbb{R}^N} k(z) dz = \frac{1}{m}.$$

We conclude in particular that for a suitable constant  $C > 0$ , we have

$$T_m[(1+|x|)^{-\mu}] \leq C(1+|x|)^{-\mu}.$$

Now, we have that

$$\pm T_m[g] \leq \|(1+|x|)^\mu g\|_{L^\infty(\mathbb{R}^N)} T_m[(1+|x|)^{-\mu}],$$

and then

$$\|(1+|x|)^\mu T[g]\|_{L^\infty(\mathbb{R}^N)} \leq C \|(1+|x|)^\mu g\|_{L^\infty(\mathbb{R}^N)}$$

as desired.  $\square$

We also have the validity of the following useful estimate.

**Lemma 2.3.** *Assume that  $g \in L^2 \cap L^\infty$ . Then the following holds: if  $\phi = T_m[g]$  then there is a  $C > 0$  such that*

$$\sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha} \leq C \|g\|_{L^\infty(\mathbb{R}^N)} \quad (2.6)$$

where  $\alpha = \min\{1, 2s\}$ .

*Proof.* Since  $\|T_m[g]\|_\infty \leq C \|g\|_\infty$ , it suffices to establish (2.6) for  $|x - y| < \frac{1}{3}$ . We have

$$|\phi(x) - \phi(y)| \leq \int_{\mathbb{R}^N} |k(z + y - x) - k(z)| dz \|g\|_\infty.$$

Now, we decompose

$$\begin{aligned} & \int_{\mathbb{R}^N} |k(z + y - x) - k(z)| dz = \\ & \int_{|z| > 3|y-x|} |k(z + y - x) - k(z)| dz + \int_{|z| < 3|y-x|} |k(z + y - x) - k(z)| dz. \end{aligned}$$

We have

$$\int_{|z| > 3|y-x|} |k(z + (y-x)) - k(z)| \leq \int_0^1 dt \int_{|z| > 3|y-x|} |\nabla k(z + t(y-x))| dz |y-x|.$$

and, since  $3|y-x| < 1$ ,

$$\int_{|z| > 3|y-x|} |\nabla k(z + t(y-x))| dz \leq C(1 + \int_{1 > |z| > 3|y-x|} \frac{dz}{|z|^{N+1-2s}}) \leq C(1 + |y-x|^{2s-1}).$$

On the other hand

$$\int_{|z| < 3|y-x|} |k(z + y - x) - k(z)| dz \leq 2 \int_{|z| < 4|y-x|} |k(z)| dz \leq C|y-x|^{2s},$$



and (2.6) readily follows.  $\square$

Next we consider the more general problem

$$(-\Delta)^s \phi + W(x)\phi = g \quad \text{in } \mathbb{R}^N \quad (2.7)$$

where  $W$  is a bounded potential.

We start with a form of the weak maximum principle.

**Lemma 2.4.** *Let us assume that*

$$\inf_{x \in \mathbb{R}^N} W(x) =: m > 0$$

*and that  $\phi \in H^{2s}(\mathbb{R}^N)$  satisfies equation (2.7) with  $g \geq 0$ . Then  $\phi \geq 0$  in  $\mathbb{R}^N$ .*

*Proof.* We use the representation for  $\phi$  as the trace of the unique solution  $\tilde{\phi} \in H$  to the problem

$$\iint_{\mathbb{R}_+^{N+1}} \nabla \tilde{\phi} \nabla \varphi y^{1-2s} + \int_{\mathbb{R}^N} W \phi \varphi = \int_{\mathbb{R}^N} g \varphi, \quad \text{for all } \varphi \in H.$$

It is easy to check that the test function  $\varphi = \phi_- = \min\{\phi, 0\}$  does indeed belong to  $H$ . We readily obtain

$$\iint_{\mathbb{R}_+^{N+1}} |\nabla \tilde{\phi}_-|^2 y^{1-2s} + \int_{\mathbb{R}^N} W \phi_-^2 = \int_{\mathbb{R}^N} g \phi_-.$$

Since  $g \geq 0$  and  $W \geq m$ , we obtain that  $\phi_- \equiv 0$ , which means precisely  $\phi \geq 0$ , as desired.  $\square$

We want to obtain a priori estimates for problems of the type (2.7) when  $W$  is not necessarily positive. Let  $\mu > \frac{N}{2}$ , and let us assume that

$$\|(1 + |x|^\mu)g\|_{L^\infty(\mathbb{R}^N)} < +\infty.$$

The assumption in  $\mu$  implies that  $g \in L^2(\mathbb{R}^N)$ .

Below, and in all what follows, we will say that  $\phi \in L^2(\mathbb{R}^N)$  solves equation (2.7) if and only if  $\phi$  solves the linear problem

$$\phi = T_m((m - W)\phi + g).$$

Similarly, we will say that

$$(-\Delta)^s \phi + W(x)\phi \geq g \quad \text{in } \mathbb{R}^N$$

if for some  $\tilde{g} \in L^2(\mathbb{R}^N)$  with  $\tilde{g} \geq g$  we have

$$\phi = T_m((m - W)\phi + \tilde{g}).$$

The next lemma provides an a priori estimate for a solution  $\phi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  of (2.7).

**Lemma 2.5.** *Let  $W$  be a continuous function, such that for  $k$  points  $q_i$   $i = 1, \dots, k$  a number  $R > 0$  and  $B = \cup_{i=1}^k B_R(q_i)$  we have*

$$\inf_{x \in \mathbb{R}^N \setminus B} W(x) =: m > 0.$$

*Then, given any number  $\frac{N}{2} < \mu < N + 2s$  there exists a constant  $C = C(\mu, k, R) > 0$  such that for any  $\phi \in H^{2s} \cap L^\infty(\mathbb{R}^N)$  and  $g$  with*

$$\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)} < +\infty$$

that satisfy equation (2.7) we have the validity of the estimate

$$\|\rho^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \leq C \left[ \|\phi\|_{L^\infty(B)} + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)} \right].$$

Here

$$\rho(x) = \sum_{i=1}^k \frac{1}{(1 + |x - q_i|)^\mu}.$$

*Proof.* We start by noticing that  $\phi$  satisfies the equation

$$(-\Delta)^s \phi + \hat{W}\phi = \hat{g}$$

where

$$\hat{g} = (m - W)\chi_B \phi, \quad \hat{W} = m\chi_B + W(1 - \chi_B).$$

Observe that

$$|\hat{g}(x)| \leq M \sum_{i=1}^k (1 + |x - q_i|)^{-\mu}, \quad M = C(\|\phi\|_{L^\infty(B)} + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)})$$

where  $C$  depends only on  $R$ ,  $k$  and  $\mu$  and

$$\inf_{x \in \mathbb{R}^N} \hat{W}(x) \geq m.$$

Now, from Lemma 2.2, since  $0 < \mu < N + 2s$  we find a solution  $\phi_0(x)$  to the problem

$$(-\Delta)^s \bar{\phi} + m\bar{\phi} = (1 + |x|)^{-\mu}$$

such that  $\bar{\phi} = O(|x|^{-\mu})$  as  $|x| \rightarrow \infty$ . Then we have that

$$((-\Delta)^s + \hat{W})(\bar{\phi}) \geq M \sum_{i=1}^k (1 + |x - q_i|)^{-\mu}$$

where

$$\bar{\phi}(x) = M \sum_{i=1}^k \phi_0(x - q_i).$$

Setting  $\psi = (\phi - \bar{\phi})$  we get

$$(-\Delta)^s \psi + \hat{W}\psi = \tilde{g} \leq 0$$

with  $\tilde{g} \in L^2$ . Using Lemma 2.4 we obtain  $\phi \leq \bar{\phi}$ . Arguing similarly for  $-\phi$ , and using the form of  $\bar{\phi}$  and  $M$ , the desired estimate immediately follows.  $\square$

Examining the proof above, we obtain immediately the following.

**Corollary 2.1.** *Let  $\rho(x)$  be defined as in the previous lemma. Assume that  $\phi \in H^{2s}(\mathbb{R}^N)$  satisfies equation (2.7) and that*

$$\inf_{x \in \mathbb{R}^N} W(x) =: m > 0.$$

*Then we have that  $\phi \in L^\infty(\mathbb{R}^N)$  and it satisfies*

$$\|\rho^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \leq C \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}. \quad (2.8)$$

A last useful fact is that if  $f, g \in L^2(\mathbb{R}^N)$  and  $W = T(f)$ ,  $Z = T(g)$  then the following holds:

$$\int_{\mathbb{R}^N} Z(-\Delta)^s W - \int_{\mathbb{R}^N} W(-\Delta)^s Z = \int_{\mathbb{R}^N} T_m[f]g - \int_{\mathbb{R}^N} T_m[g]f = 0,$$

the latter fact since the kernel  $k$  is radially symmetric.

## 3. FORMULATION OF THE PROBLEM: THE ANSATZ

By a solution of the problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0 \quad \text{in } \mathbb{R}^N$$

we mean a  $u \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  such that the above equation is satisfied. Let us observe that it suffices to solve

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u_+^p = 0 \quad \text{in } \mathbb{R}^N \quad (3.1)$$

where  $u_+ = \max\{u, 0\}$ . In fact, if  $u$  solves (3.1) then

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u \geq 0 \quad \text{in } \mathbb{R}^N$$

and, as a consequence to Lemma 2.4,  $u \geq 0$ .

After absorbing  $\varepsilon$  by scaling, the equation takes the form

$$(-\Delta)^s v + V(\varepsilon x)v - v_+^p = 0 \quad \text{in } \mathbb{R}^N \quad (3.2)$$

Let us consider points  $\xi_1, \dots, \xi_k \in \mathbb{R}^N$  and designate

$$q_i = \varepsilon^{-1}\xi_i, \quad q = (q_1, \dots, q_k).$$

Given numbers  $\delta > 0$  small and  $R > 0$  large, we define the configuration space  $\Gamma$  for the points  $q_i$  as

$$\Gamma := \{q = (q_1, \dots, q_k) \mid R \leq \max_{i \neq j} |q_i - q_j|, \quad \max_i |q_i| \leq \delta^{-1}\varepsilon^{-1}\}. \quad (3.3)$$

We look for a solution with concentration behavior near each  $\xi_j$ . Letting  $\tilde{v}(x) = v(x + \xi_j)$  translating the origin to  $q_j$ , Equation (3.2) reads

$$(-\Delta)^s \tilde{v} + V(\xi_j + \varepsilon x)\tilde{v} - \tilde{v}_+^p = 0 \quad \text{in } \mathbb{R}^N.$$

Letting formally  $\varepsilon \rightarrow 0$  we are left with the equation

$$(-\Delta)^s \tilde{v} + \lambda_j \tilde{v} - \tilde{v}_+^p = 0 \quad \text{in } \mathbb{R}^N, \quad \lambda_j = V(\xi_j).$$

So we ask that  $v(x) \approx w_{\lambda_j}(x - q_j)$  near  $q_j$ . We consider the sum of these functions as a first approximation. Thus, we look for a solution  $v$  of (3.2) of the form

$$v = W_q + \phi, \quad W_q(x) = \sum_{j=1}^k w_j(x), \quad w_j(x) = w_{\lambda_j}(x - q_j), \quad \lambda_j = V(\xi_j),$$

where  $\phi$  is a small function, disappearing as  $\varepsilon \rightarrow 0$ . In terms of  $\phi$ , Equation (3.2) becomes

$$(-\Delta)^s \phi + V(\varepsilon x)\phi - pW_q^{p-1}\phi = E + N(\phi) \quad \text{in } \mathbb{R}^N \quad (3.4)$$

where

$$\begin{aligned} N(\phi) &:= (W_q + \phi)_+^p - pW_q^{p-1}\phi - W_q^p, \\ E &:= \sum_{j=1}^k (\lambda_j - V(\varepsilon x))w_j + \left( \sum_{j=1}^k w_j \right)^p - \sum_{j=1}^k w_j^p. \end{aligned} \quad (3.5)$$

Rather than solving Problem (3.4) directly, we consider first a projected version of it. Let us consider the functions

$$Z_{ij}(x) := \partial_j w_i(x)$$

and the problem of finding  $\phi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  such that for certain constants  $c_{ij}$

$$(-\Delta)^s \phi + V(\varepsilon x) \phi - pW_q^{p-1} \phi = E + N(\phi) + \sum_{i=1}^k \sum_{j=1}^N c_{ij} Z_{ij}, \quad (3.6)$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j. \quad (3.7)$$

Let  $\mathcal{Z}$  be the linear space spanned by the functions  $Z_{ij}$ , so that equation (3.6) is equivalent to

$$(-\Delta)^s \phi + V(\varepsilon x) \phi - pW_q^{p-1} \phi - E - N(\phi) \in \mathcal{Z}.$$

On the other hand, for all  $\varepsilon$  sufficiently small, the functions  $Z_{ij}$  are linearly independent, hence the constants  $c_{ij}$  have unique, computable expressions in terms of  $\phi$ . We will prove that Problem (3.6)-(3.7) has a unique small solution  $\phi = \Phi(q)$ . In that way we will get a solution to the full problem (3.4) if we can find a value of  $q$  such that  $c_{ij}(\Phi(q)) = 0$  for all  $i, j$ . In order to build  $\Phi(q)$  we need a theory of solvability for associated linear operator in suitable spaces. This is what we develop in the next section.

#### 4. LINEAR THEORY

We consider the linear problem of finding  $\phi \in H^{2s}(\mathbb{R}^N)$  such that for certain constants  $c_{ij}$  we have

$$(-\Delta)^s \phi + V(\varepsilon x) \phi - pW_q^{p-1}(x) \phi + g(x) = \sum_{i=1}^N \sum_{j=1}^k c_{ij} Z_{ij} \quad (4.1)$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j. \quad (4.2)$$

The constants  $c_{ij}$  are uniquely determined in terms of  $\phi$  and  $g$  when  $\varepsilon$  is sufficiently small, from the linear system

$$\sum_{i,j} c_{ij} \int_{\mathbb{R}^N} Z_{ij} Z_{lk} = \int_{\mathbb{R}^N} Z_{lk} [(-\Delta)^s \phi + V(\varepsilon x) \phi - pW_q^{p-1}(x) \phi + g]. \quad (4.3)$$

Taking into account that

$$\int_{\mathbb{R}^N} Z_{lk} (-\Delta)^s \phi = \int_{\mathbb{R}^N} \phi (-\Delta)^s Z_{lk} = \int_{\mathbb{R}^N} (pw_l^{p-1} - \lambda_l) Z_{lk} \phi,$$

we find

$$c_{ij} \int_{\mathbb{R}^N} Z_{ij} Z_{lk} = \int_{\mathbb{R}^N} g Z_{lk} + (pw_l^{p-1} - pW_q^{p-1} + V(\varepsilon x) - \lambda_l) \int_{\mathbb{R}^N} Z_{lk} \phi. \quad (4.4)$$

On the other hand, we check that

$$\int_{\mathbb{R}^N} Z_{ij} Z_{lk} = \alpha_l \delta_{ijkl} + O(d^{-N})$$

where the numbers  $\alpha_l$  are positive, and independent of  $\varepsilon$ , and

$$d = \min\{|q_i - q_j| \mid i \neq j\} \gg 1.$$

Then, we see that relations (4.4) define a uniquely solvable (nearly diagonal) linear system, provided that  $\varepsilon$  is sufficiently small. We assume this last fact in what follows, and hence that the numbers  $c_{ij} = c_{ij}(\phi, g)$  are defined by relations (4.4).

Moreover, we have that

$$|(pw_l^{p-1} - pW_q^{p-1} + V(\varepsilon x) - \lambda_l) Z_{lk}(x)| \leq C(R^{-N} + \varepsilon|x - q_j|)(1 + |x - q_j|)^{-N-s}$$

and then from expression (4.4) we obtain the following estimate.

**Lemma 4.1.** *The numbers  $c_{ij}$  in (4.1) satisfy:*

$$c_{ij} = \frac{1}{\alpha_i} \int_{\mathbb{R}^N} g Z_{ij} + \theta_{ij}.$$

where

$$|\theta_{ij}| \leq C(\varepsilon + d^{-N}) [\|\phi\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^2(\mathbb{R}^N)}].$$

In the rest of this section we shall build a solution to Problem (4.1)-(4.2).

**Proposition 4.1.** *Given  $k \geq 1$ ,  $\frac{N}{2} < \mu < N + 2s$ ,  $C > 0$ , there exist positive numbers  $d_0, \varepsilon_0, C$  such that for any points  $q_1, \dots, q_k$  and any  $\varepsilon$  with*

$$\sum_{i=1}^k |q_i| \leq \frac{C}{\varepsilon}, \quad R := \min\{|q_i - q_j| / i \neq j\} > R_0, \quad 0 < \varepsilon < \varepsilon_0$$

*there exists a solution  $\phi = T[g]$  of (4.1)-(4.2) that defines a linear operator of  $g$ , provided that*

$$\|\rho(x)^{-1}g\|_{L^\infty(\mathbb{R}^N)} < +\infty, \quad \rho(x) = \sum_{j=1}^k \frac{1}{(1 + |x - q_j|)^\mu}.$$

Besides

$$\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \leq C\|\rho(x)^{-1}g\|_{L^\infty(\mathbb{R}^N)}.$$

To prove this result we require several steps. We begin with corresponding a priori estimates.

**Lemma 4.2.** *Under the conditions of Proposition 4.1, there exists a  $C > 0$  such that for any solution of (4.1)-(4.2) with  $\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} < +\infty$  we have the validity of the a priori estimate*

$$\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \leq C\|\rho(x)^{-1}g\|_{L^\infty(\mathbb{R}^N)}.$$

*Proof.* Let us assume the a priori estimate does not hold, namely there are sequences  $\varepsilon_n \rightarrow 0$ ,  $q_{jn}$ ,  $j = 1, \dots, k$ , with

$$\min\{|q_{in} - q_{jn}| / i \neq j\} \rightarrow \infty$$

and  $\phi_n, g_n$  with

$$\|\rho_n(x)^{-1}\phi_n\|_{L^\infty(\mathbb{R}^N)} = 1, \quad \|\rho_n(x)^{-1}g_n\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0,$$

where

$$\rho_n(x) = \sum_{j=1}^k \frac{1}{(1 + |x - q_{jn}|)^\mu},$$

with  $\phi_n, g_n$  satisfying (4.1)-(4.2). We claim that for any fixed  $R > 0$  we have that

$$\sum_{j=1}^k \|\phi_n\|_{L^\infty(B_R(q_{jn}))} \rightarrow 0. \tag{4.5}$$

Indeed, assume that for a fixed  $j$  we have that  $\|\phi_n\|_{L^\infty(B_R(q_{jn}))} \geq \gamma > 0$ . Let us set  $\bar{\phi}_n(x) = \phi_n(q_{jn} + x)$ . We also assume that  $\lambda_j^n = V(q_{jn}) \rightarrow \lambda > 0$  and

$$(-\Delta)^s \bar{\phi}_n + V(q_{jn} + \varepsilon_n x) \bar{\phi}_n + p(w_{\lambda_j^n}(x) + \theta_n(x))^{p-1} \bar{\phi}_n = \bar{g}_n$$

where

$$\bar{g}_n(x) = g_n(q_{jn} + x) - \sum_{l=1}^k \sum_{i=1}^n c_{ln}^i \partial_i w_{\lambda_j^n}(q_{jn} - q'_{ln} + x).$$

We observe that  $\bar{g}_n(x) \rightarrow 0$  uniformly on compact sets. From the uniform Hölder estimates (2.6), we also obtain equicontinuity of the sequence  $\bar{\phi}_n$ . Thus, passing to a subsequence, we may assume that  $\bar{\phi}_n$  converges, uniformly on compact sets, to a bounded function  $\bar{\phi}$  which satisfies  $\|\bar{\phi}\|_{L^\infty(B_R(0))} \geq \gamma$ . In addition, we have that

$$\|(1 + |x|)^\mu \bar{\phi}\|_{L^\infty(\mathbb{R}^N)} \leq 1$$

and that  $\bar{\phi}$  solves the equation

$$(-\Delta)^s \bar{\phi} + \bar{\lambda} \bar{\phi} + p w_{\bar{\lambda}}^{p-1} \bar{\phi} = 0$$

Let us notice that  $\bar{\phi} \in L^2(\mathbb{R}^N)$ , and hence the nondegeneracy result in [15] applies to yield that  $\bar{\phi}$  must be a linear combination of the partial derivatives  $\partial_i w_{\bar{\lambda}}$ . But the orthogonality conditions pass to the limit, and yield

$$\int_{\mathbb{R}^N} \partial_i w_{\bar{\lambda}} \bar{\phi} = 0 \quad \text{for all } i = 1, \dots, N.$$

Thus, necessarily  $\bar{\phi} = 0$ . We have obtained a contradiction that proves the validity of (4.5). This and the a priori estimate in Lemma 2.5 shows that also,  $\|\rho_n(x)^{-1} \phi_n\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$ , again a contradiction that proves the desired result.  $\square$

Next we construct a solution to problem (4.1)-(4.2). To do so, we consider first the auxiliary problem

$$(-\Delta)^s \phi + V\phi = g + \sum_{i=1}^k \sum_{j=1}^N c_{ij} Z_{ij}, \quad (4.6)$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j. \quad (4.7)$$

where  $V$  is our bounded, continuous potential with

$$\inf_{\mathbb{R}^N} V = m > 0$$

**Lemma 4.3.** *For each  $g$  with  $\|\rho^{-1}g\|_\infty < +\infty$ , there exists a unique solution of Problem (4.1)-(4.2),  $\phi =: A[g] \in H^{2s}(\mathbb{R}^N)$ . This solution satisfies*

$$\|\rho^{-1}A[g]\|_{L^\infty(\mathbb{R}^N)} \leq C\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}. \quad (4.8)$$

*Proof.* First we write a variational formulation for this problem. Let  $X$  be the closed subspace of  $H$  defined as

$$X = \{\tilde{\phi} \in H \mid \int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j\}$$

Then, given  $g \in L^2$ , we consider the problem of finding a  $\tilde{\phi} \in X$  such that

$$\langle \tilde{\phi}, \tilde{\psi} \rangle := \iint_{\mathbb{R}_+^{N+1}} \nabla \tilde{\phi} \nabla \tilde{\psi} y^{1-2s} + \int_{\mathbb{R}^N} V \phi \psi = \int_{\mathbb{R}^N} g \psi \quad \text{for all } \psi \in X. \quad (4.9)$$

We observe that  $\langle \cdot, \cdot \rangle$  defines an inner product in  $X$  equivalent to that of  $H$ . Thus existence and uniqueness of a solution follows from Riesz's theorem. Moreover, we see that

$$\|\phi\|_{L^2(\mathbb{R}^N)} \leq C\|g\|_{L^2(\mathbb{R}^N)}.$$

Next we check that this produces a solution in strong sense. Let  $\mathcal{Z}$  be the space spanned by the functions  $Z_{ij}$ . We denote by  $\Pi[g]$  the  $L^2(\mathbb{R}^N)$  orthogonal projection of  $g$  onto  $\mathcal{Z}$  and by  $\tilde{\Pi}[g]$  its natural  $s$ -harmonic extension. For a function  $\tilde{\varphi} \in H$  let us write

$$\tilde{\psi} = \tilde{\varphi} - \tilde{\Pi}[\varphi]$$

so that  $\tilde{\psi} \in X$ . Substituting this  $\tilde{\psi}$  into (4.9) we obtain

$$\begin{aligned} & \iint_{\mathbb{R}_+^{N+1}} \nabla \tilde{\phi} \nabla \tilde{\varphi} y^{1-2s} + \int_{\mathbb{R}^N} V \phi \varphi = \\ & \int_{\mathbb{R}^N} g \varphi + \int_{\mathbb{R}^N} [V \phi - g] \Pi[\varphi] + \int_{\mathbb{R}^N} \phi (-\Delta)^s \Pi[\varphi]. \end{aligned}$$

Here we have used that  $\tilde{\Pi}[\varphi]$  is regular and

$$\iint_{\mathbb{R}_+^{N+1}} \nabla \phi \nabla \tilde{\Pi}[\varphi] y^{1-2s} = \int_{\mathbb{R}^N} \phi (-\Delta)^s \Pi[\varphi].$$

Let us observe that for  $f \in L^2(\mathbb{R}^N)$  the functional

$$\ell(f) = \int_{\mathbb{R}^N} \phi (-\Delta)^s \Pi[f]$$

satisfies

$$|\ell(f)| \leq C\|\phi\|_{L^2(\mathbb{R}^N)}\|\psi\|_{L^2(\mathbb{R}^N)},$$

hence there is an  $h(\phi) \in L^2(\mathbb{R}^N)$  such that

$$\ell(\psi) = \int_{\mathbb{R}^N} h \psi.$$

If  $\phi$  was a priori known to be in  $H^{2s}(\mathbb{R}^N)$  we would have precisely that

$$h(\phi) = \Pi[(-\Delta)^s \phi].$$

Since  $\Pi$  is a self-adjoint operator in  $L^2(\mathbb{R}^N)$  we then find that

$$\iint_{\mathbb{R}_+^{N+1}} \nabla \tilde{\phi} \nabla \tilde{\varphi} y^{1-2s} + \int_{\mathbb{R}^N} V \phi \varphi = \int_{\mathbb{R}^N} \bar{g} \varphi$$

where

$$\bar{g} = g + \Pi[V \phi - g] + h(\phi).$$

Since  $\bar{g} \in L^2(\mathbb{R}^N)$ , it follows then that  $\phi \in H^{2s}(\mathbb{R}^N)$  and it satisfies

$$(-\Delta)^s \phi + V \phi - g = \Pi[(-\Delta)^s \phi + V \phi - g] \in \mathcal{Z},$$

hence equations (4.6)-(4.7) are satisfied. To establish estimate (4.8), we use just Corollary 2.1, observing that

$$\begin{aligned} \|\rho^{-1} \Pi[(-\Delta)^s \phi + V \phi - g]\|_{L^\infty(\mathbb{R}^N)} & \leq C(\|\phi\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^2(\mathbb{R}^N)}) \leq \\ & C\|g\|_{L^2(\mathbb{R}^N)} \leq \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)}. \end{aligned}$$

The proof is concluded.  $\square$

**Proof of Proposition 4.1.** Let us solve now Problem (4.1)-(4.2). Let  $Y$  be the Banach space

$$Y := \{\phi \in C(\mathbb{R}^N) / \|\phi\|_Y := \|\rho^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} < +\infty\} \quad (4.10)$$

Let  $A$  be the operator defined in Lemma 4.3. Then we have a solution to Problem (4.1)-(4.2) if we solve

$$\phi - A[pW_q^{p-1}\phi] = A[g], \quad \phi \in Y. \quad (4.11)$$

We claim that

$$B[\phi] := A[pW_q^{p-1}\phi]$$

defines a compact operator in  $Y$ . Indeed. Let us assume that  $\phi_n$  is a bounded sequence in  $Y$ . We observe that for some  $\sigma > 0$  we have

$$|W_q^{p-1}\phi_n| \leq C\|\phi_n\|_Y \rho^{1+\sigma}.$$

If  $\sigma$  is sufficiently small, it follows that  $f_n := B[\phi_n]$  satisfies

$$|\rho^{-1}f_n| \leq C\rho^\sigma$$

Besides, since  $f_n = T_m((V-m)f_n + g_n)$  we use estimate (2.6) to get that for some  $\alpha > 0$

$$\sup_{x \neq y} \frac{|f_n(x) - f_n(y)|}{|x - y|^\alpha} \leq C.$$

Arzela's theorem then yields the existence of a subsequence of  $f_n$  which we label the same way, that converges uniformly on compact sets to a continuous function  $f$  with

$$|\rho^{-1}f| \leq C\rho^\sigma.$$

Let  $R > 0$  be a large number. Then we estimate

$$\|\rho^{-1}(f_n - f)\|_{L^\infty(\mathbb{R}^N)} \leq \|\rho^{-1}(f_n - f)\|_{L^\infty(B_R(0))} + C \max_{|x|>R} \rho^\sigma(x).$$

Since

$$\max_{|x|>R} \rho^\sigma(x) \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

we conclude then that  $\|f_n - f\|_\infty \rightarrow 0$  and the claim is proven.

Finally, the a priori estimate tells us that for  $g = 0$ , equation (4.11) has only the trivial solution. The desired result follows at once from Fredholm's alternative.  $\square$

We conclude this section by analyzing the differentiability with respect to the parameter  $q$  of the solution  $\phi = T_q[g]$  of (4.1)-(4.2). As in the proof above we let  $Y$  be the space in (4.10), so that  $T_q \in \mathcal{L}(Y)$

**Lemma 4.4.** *The map  $q \mapsto T_q$  is continuously differentiable, and for some  $C > 0$ ,*

$$\|\partial_q T_q\|_{\mathcal{L}(Y)} \leq C \quad (4.12)$$

for all  $q$  satisfying constraints (3.3).

*Proof.* Let us write  $q = (q_1, \dots, q_k)$ ,  $q_i = (q_{i1}, \dots, q_{iN})$ ,  $\phi = T_q[g]$ , and (formally)

$$\psi = \partial_{q_{ij}} T_q[g], \quad d_{lk} = \partial_{q_{ij}} c_{lk}.$$

Then, by differentiation of equations (4.1)-(4.2), we get

$$(-\Delta)^s \psi + V(\varepsilon x) \psi - pW_q^{p-1} \psi = p \partial_{q_{ij}} W_q^{p-1} \phi + \sum_{l,k} c_{lk} \partial_{q_{ij}} Z_{lk} + \sum_{l,k} d_{lk} Z_{lk}, \quad (4.13)$$



$$\int_{\mathbb{R}^N} \psi Z_{lk} = - \int_{\mathbb{R}^N} \phi \partial_{q_{ij}} Z_{lk} \quad \text{for all } l, k. \quad (4.14)$$

We let

$$\tilde{\psi} = \psi - \Pi[\psi]$$

where, as before,  $\Pi[\psi]$  denotes the orthogonal projection of  $\psi$  onto the space spanned by the  $Z_{lk}$ . Writing

$$\Pi[\psi] = \sum_{l,k} \alpha_{lk} Z_{lk} \quad (4.15)$$

and relations (4.14) as

$$\int_{\mathbb{R}^N} \Pi[\psi] Z_{lk} = - \int_{\mathbb{R}^N} \phi \partial_{q_{ij}} Z_{lk} \quad \text{for all } l, k, \quad (4.16)$$

we get

$$|\alpha_{lk}| \leq C \|\phi\|_Y \leq C \|g\|_Y. \quad (4.17)$$

From (4.13) we have then that

$$(-\Delta)^s \tilde{\psi} + V(\varepsilon x) \tilde{\psi} - p W_q^{p-1} \tilde{\psi} = \tilde{g} + \sum_{l,k} d_{lk} Z_{lk}, \quad (4.18)$$

or  $\tilde{\psi} = T_q[\tilde{g}]$  where

$$\tilde{g} = p \partial_{q_{ij}} W_q^{p-1} \phi + \sum_{l,k} c_{lk} \partial_{q_{ij}} Z_{lk} - [(-\Delta)^s + V(\varepsilon x) - p W_q^{p-1}] \Pi[\psi]. \quad (4.19)$$

Then we see that

$$\|\tilde{\psi}\|_Y \leq C \|\tilde{g}\|_Y.$$

Using (4.17) and Lemma 4.1, we see also that

$$\|\tilde{g}\|_Y \leq C \|g\|_Y, \quad \|\Pi[\psi]\| \leq C \|g\|_Y$$

and thus

$$\|\psi\| \leq C \|g\|_Y. \quad (4.20)$$

Let us consider now, rigorously, the unique  $\psi = \tilde{\psi} + \Pi[\psi]$  that satisfies equations (4.14) and (4.19). We want to show that indeed

$$\psi = \partial_{q_{ij}} T_q[g].$$

To do so,  $q_i^t = q_i + t e_j$  where  $e_j$  is the  $j$ -th element of the canonical basis of  $\mathbb{R}^N$ , and set

$$q^t = (q_1, \dots, q_{i-1}, q_i^t, \dots, q_k).$$

For a function  $f(q)$  we denote

$$D_{ij}^t f = t^{-1} (f(q^t) - f(q))$$

we also set

$$\phi^t := T_{q^t}[g], \quad D_{ij}^t T_q[g] =: \psi^t = \tilde{\psi}^t + \Pi[\tilde{\psi}^t]$$

so that

$$(-\Delta)^s \tilde{\psi}^t + V(\varepsilon x) \tilde{\psi}^t - p W_q^{p-1} \tilde{\psi}^t = \tilde{g}^t + \sum_{l,k} d_{lk}^t Z_{lk},$$

where

$$\tilde{g}^t = p D_{ij}^t [W_q^{p-1}] \phi + \sum_{l,k} c_{lk} D_{ij}^t Z_{lk} - [(-\Delta)^s + V(\varepsilon x) - p W_q^{p-1}] \Pi[\psi^t],$$

$$d_{lk}^t = D_{ij}^t c_{lk}$$

and

$$\Pi[\psi^t] = \sum_{l,k} \alpha_{lk}^t Z_{lk},$$

where the constants  $\alpha_{lk}^t$  are determined by the relations

$$\int_{\mathbb{R}^N} \Pi[\psi^t] Z_{lk} = - \int_{\mathbb{R}^N} \phi D_{ij}^t Z_{lk}.$$

Comparing these relations with (4.15), (4.16), (4.18) defining  $\psi$ , we obtain that

$$\lim_{t \rightarrow 0} \|\psi^t - \psi\|_Y = 0$$

which by definition tells us  $\psi = \partial_{q_{ij}} T_q[g]$ . The continuous dependence in  $q$  is clear from that of the data in the definition of  $\psi$ . Estimate (4.12) follows from (4.20). The proof is concluded.  $\square$

## 5. SOLVING THE NONLINEAR PROJECTED PROBLEM

In this section we solve the nonlinear projected problem

$$(-\Delta)^s \phi + V(\varepsilon x) \phi - p W_q^{p-1} \phi = E + N(\phi) + \sum_{i=1}^k \sum_{j=1}^N c_{ij} Z_{ij}, \quad (5.1)$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j. \quad (5.2)$$

We have the following result.

**Proposition 5.1.** *Assuming that  $\|E\|_Y$  is sufficiently small problem (5.1)-(5.2) has a unique small solution  $\phi = \Phi(q)$  with*

$$\|\Phi(q)\|_Y \leq C \|E\|_Y$$

*The map  $q \mapsto \Phi(q)$  is of class  $C^1$ , and for some  $C > 0$*

$$\|\partial_q \Phi(q)\|_Y \leq C [\|E\|_Y + \|\partial_q E\|_Y]. \quad (5.3)$$

*for all  $q$  satisfying constraints (3.3).*

*Proof.* Problem (5.1)-(5.2) can be written as the fixed point problem

$$\phi = T_q(E + N(\phi)) =: K_q(\phi), \quad \phi \in Y. \quad (5.4)$$

Let

$$B = \{\phi \in Y \mid \|\phi\|_Y \leq \rho\}.$$

If  $\phi \in B$  we have that

$$|N(\phi)| \leq C |\phi|^\beta, \quad \beta = \min\{p, 2\}.$$

and hence

$$\|N(\phi)\|_Y \leq C \|\phi\|^2$$

It follows that

$$\|K_q(\phi)\|_Y \leq C_0 [\|E\| + \rho^2]$$

for a number  $C_0$ , uniform in  $q$  satisfying (3.3). Let us assume

$$\rho := 2C_0 \|E\|, \quad \|E\| \leq \frac{1}{2C_0}.$$

Then

$$\|K_q(\phi)\|_Y \leq C_0 \left[ \frac{1}{2C_0} \rho + \rho^2 \right] \leq \rho$$

so that  $K_q(B) \subset B$ . Now, we observe that

$$|N(\phi_1) - N(\phi_2)| \leq C[|\phi|^{\beta-1} + |\phi|^{\beta-1}]|\phi_1 - \phi_2|$$

and hence

$$\|N(\phi_1) - N(\phi_2)\|_Y \leq C\rho^{\beta-1}\|\phi_1 - \phi_2\|_Y$$

and

$$\|K_q(\phi_1) - K_q(\phi_2)\| \leq C\rho^{\beta-1}\|\phi_1 - \phi_2\|_Y.$$

Reducing  $\rho$  if necessary, we obtain that  $K_q$  is a contraction mapping and hence has a unique solution of equation (5.4) exists in  $B$ . We denote it as  $\phi = \Phi(q)$ . We prove next that  $\Phi$  defines a  $C^1$  function of  $q$ . Let

$$M(\phi, q) := \phi - T_q(E + N(\phi))$$

Let  $\phi_0 = \Phi(q_0)$ . Then  $M(\phi_0, q_0) = 0$ . On the other hand,

$$\partial_\phi M(\phi, q)[\psi] = \psi - T_q(N'(\phi)\psi)$$

where  $N'(\phi) = p[(W + \phi)^{p-1} - W^{p-1}]$ , so that

$$\|N'(\phi)\psi\|_Y \leq C\rho^{\beta-1}\|\psi\|_Y$$

If  $\rho$  is sufficiently small we have then that  $D_\phi M(\phi_0, q_0)$  is an invertible operator, with uniformly bounded inverse. Besides

$$\partial_q M(\phi, q) = (\partial_q T_q)(E + N(\phi)) + T_q(\partial_q E + \partial_q N(\phi))$$

Both partial derivatives are continuous in their arguments. The implicit function applies in a small neighborhood of  $(\phi_0, q_0)$  to yield existence and uniqueness of a function  $\phi = \phi(q)$  with  $\phi(q_0) = \phi_0$  defined near  $q_0$  with  $M(\phi(q), q) = 0$ . Besides,  $\phi(q)$  is of class  $C^1$ . But, by uniqueness, we must have  $\phi(q) = \Phi(q)$ . Finally, we see that

$$\partial_q \Phi(q) = -D_\phi M(\Phi(q), q)^{-1} [(\partial_q T_q)(E + N(\Phi(q))) + T_q(\partial_q E + \partial_q N(\Phi(q)))]$$

$$\partial_q N(\phi) = p[(W + \phi)^{p-1} - pW^{p-1} - (p-1)W^{p-2}\phi]\partial_q W$$

and hence

$$\|(\partial_q N)(\Phi(q))\|_Y \leq C\|\Phi(q)\|_Y^\beta \leq C\|E\|_Y^\beta$$

From here, the above expressions and the bound of Lemma 4.4 we finally get the validity of Estimate (5.3).  $\square$

**5.1. An estimate of the error.** Here we provide an estimate of the error  $E$  defined in (3.5),

$$E := \sum_{j=1}^k (\lambda_j - V(\varepsilon x))w_j + \left( \sum_{j=1}^k w_j \right)^p - \sum_{j=1}^k w_j^p$$

in the norm  $\|\cdot\|_Y$ . Here we need to take  $\mu \in (\frac{N}{2}, \frac{N+2s}{2})$ . We denote

$$R = \min_{i \neq j} |q_i - q_j| \gg 1.$$

The first term in  $E$  can be easily estimated as

$$|\rho^{-1}(x) \sum_{j=1}^k (\lambda_j - V(\varepsilon x))w_j| \leq C\varepsilon^{\min(2s, 1)}.$$

To estimate the interaction term in  $E$ , we divide the  $\mathbb{R}^N$  into the  $k$  sub-domains

$$\Omega_j = \{w_j \geq w_i, \forall i \neq j\}, \quad j = 1, \dots, k.$$

In  $\Omega_j$ , we have

$$\begin{aligned} |(\sum_{j=1}^k w_j)^p - \sum_{j=1}^k w_j^p| &\leq C w_j^{p-1} \sum_{i \neq j} \frac{1}{|x - q_i|^{N+2s}} \\ &\leq C \frac{1}{(1 + |x - q_j|)^{(N+2s)(p-1)+\mu}} \sum_{i \neq j} \frac{1}{|q_j - q_i|^{N+2s-\mu}} \\ &\leq C \rho(x) R^{\mu-N-2s} \end{aligned}$$

In summary, we conclude that

$$\|E\|_Y \leq C\varepsilon^{2s} + CR^{\mu-N-2s} \quad (5.5)$$

As a consequence of Proposition 5.1 and the estimate (5.5), we obtain that

$$\|\Phi(q)\|_Y \leq C\varepsilon^{\min(2s,1)} + CR^{\mu-N-2s}.$$

Let us now take

$$\tau = C\varepsilon^{\min(2s,1)} + CR^{\mu-N-2s}$$

## 6. THE VARIATIONAL REDUCTION

We will use the above introduced ingredients to find existence results for the equation

$$(-\Delta)^s v + V(\varepsilon x)v - v_+^p = 0 \quad (6.1)$$

An energy whose Euler-Lagrange equation corresponds formally to (6.1) is given by

$$J_\varepsilon(\tilde{v}) := \frac{1}{2} \int_{\mathbb{R}^N} v(-\Delta)^s v + V(\varepsilon x)v^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} V(\varepsilon x)v^2$$

We want to find a solution of (6.1) with the form

$$v = v_q := W_q + \Phi(q)$$

where  $\Phi(q)$  is the function in Proposition 5.1. We observe that

$$(-\Delta)^s v_q + V(\varepsilon x)v_q - (v_q)_+^p = \sum_{i,j} c_{ij} Z_{ij} \quad (6.2)$$

hence what we need is to find points  $q$  such that  $c_{ij} = 0$  for all  $i, j$ . This problem can be formulated variationally as follows

**Lemma 6.1.** *Let us consider the function of points  $q = (q_1, \dots, q_k)$  given by*

$$I(q) := J_\varepsilon(W_q + \Phi(q)).$$

*where  $W_q + \Phi(q)$  is the unique  $s$ -harmonic extension of  $W_q + \Phi(q)$ . Then in (6.2), we have  $c_{ij} = 0$  for all  $i, j$  if and only if*

$$\partial_q I(q) = 0.$$

*Proof.* Let us write  $v_q = W_q + \phi(q)$ . We observe that

$$\begin{aligned} \partial_{q_{ij}} I(q) &= \int_{\mathbb{R}_+^{N+1}} \nabla \tilde{v}_q \nabla (\partial_{q_{ij}} \tilde{v}_q) y^{1-2s} + \int_{\mathbb{R}^N} V(\varepsilon x) v_q \partial_{q_{ij}} v_q - \int_{\mathbb{R}^N} (v_q)_+^{p-1} \partial_{q_{ij}} v_q = \\ &= \int_{\mathbb{R}^N} [(-\Delta)^s v_q + V(\varepsilon x) v_q - (v_q)_+^p] \partial_{q_{ij}} v_q = \sum_{k,l} c_{kl} \int_{\mathbb{R}^N} Z_{kl} \partial_{q_{ij}} v_q. \end{aligned} \quad (6.3)$$

We observe that

$$\partial_{q_{ij}} v_q = -Z_{ij} + O(\varepsilon \rho) + \partial_{q_{ij}} \Phi(q)$$

Since, according to Proposition (5.1)

$$\|\partial_q \Phi(q)\|_Y = O(\|E\|_Y + \|\partial_q E\|_Y)$$

and this quantity gets smaller as the number  $\delta$  in (3.3) is reduced, and the functions  $Z_{kl}$  are linearly independent (in fact nearly orthogonal in  $L^2$ ), it follows that the quantity in (6.3) equals zero for all  $i, j$  if and only if  $c_{ij} = 0$  for all  $i, j$ . The proof is concluded.  $\square$

Our task is therefore to find critical points of the functional  $I(q)$ . Useful to this end is to achieve expansions of the energy in special situations.

**Lemma 6.2.** *Assume that the numbers  $\delta$  and  $R$  in the definition of  $\Gamma$  in (3.3) is taken so small that*

$$\|E\|_Y + \|\partial_q E\| \leq \tau \ll 1.$$

*Then*

$$I_\varepsilon(q) = J_\varepsilon(W_q) + O(\tau^2)$$

*and*

$$\partial_q I_\varepsilon(q) = \partial_q J_\varepsilon(W_q) + O(\tau^2)$$

*uniformly on points  $q$  in  $\Gamma$ .*

*Proof.* Let us estimate

$$I(q) = J_\varepsilon(v_q), \quad v_q = W_q + \Phi(q).$$

We have that

$$I(\xi) = \frac{1}{2} \int_{\mathbb{R}^N} v_q (-\Delta)^s v_q + V v_q^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} v_q^{p+1}$$

Thus we can expand

$$\begin{aligned} I(q) &= J_\varepsilon(W_q) + \int_{\mathbb{R}^N} \Phi [(-\Delta)^s v_q + V v_q - v_q^p] + \frac{1}{2} \int_{\mathbb{R}^N} \Phi (-\Delta)^s \Phi + V \Phi^2 \\ &\quad - \frac{1}{p+1} \int_{\mathbb{R}^N} [(W_q + \Phi)^{p+1} - W_q^{p+1} - (p+1) W_q^p \Phi] \end{aligned}$$

Since,  $\|E\|_Y \leq \tau$  then  $\|\Phi\|_Y = O(\tau)$ , and from the equation satisfied by  $\Phi$ , also  $\|(-\Delta)^s \Phi\|_Y = O(\tau)$ . This implies

$$\left| \frac{1}{2} \int_{\mathbb{R}^N} \Phi (-\Delta)^s \Phi + V \Phi^2 \right| \leq C \int_{\mathbb{R}^N} \rho^{2\mu} \tau^2 \leq C \tau^2$$

and

$$\left| \int_{\mathbb{R}^N} [(W_q + \Phi)^{p+1} - W_q^{p+1} - (p+1) W_q^p \Phi] \right| \leq C \int_{\mathbb{R}^N} \rho^{2\mu} \tau^2 \leq C \tau^2$$

Here we have used the fact that  $\mu \in (\frac{N}{2}, \frac{N+2s}{2})$ .

On the other hand the second term in the above expansion equals 0, since by definition

$$(-\Delta)^s v_q + V v_q - v_q^p \in \mathcal{Z}$$

and  $\Phi$  is  $L^2$ -orthogonal to that space. We arrive to the conclusion that

$$I(q) = J_\varepsilon(W_q) + O(\tau^2)$$

uniformly for  $q$  in a bounded set. By differentiation we also have that

$$\begin{aligned} \partial_q I(q) &= \partial_q J_\varepsilon(W_q) + \int_{\mathbb{R}^N} \partial_q \Phi (-\Delta)^s \Phi + V \Phi \partial_q \Phi + \\ &\quad \int_{\mathbb{R}^N} [(W_q + \Phi)^p - W_q^p - p W_q^{p-1} \Phi] \partial_q W_q + [(W_q + \Phi)^p - W_q^p] \partial_q \Phi. \end{aligned}$$

Since we also have  $\|\partial_q \Phi\|_Y = O(\tau)$ , then the second and third term above are of size  $O(\varepsilon^2)$ . Thus,

$$\partial_q I(q) = \partial_q J_\varepsilon(W_q) + O(\rho^2).$$

uniformly on  $q \in \Gamma$  and the proof is complete.  $\square$

Next we estimate  $J_\varepsilon(W_q)$  and  $\partial_q J_\varepsilon(W_q)$ . We begin with the simpler case  $k = 1$ . Here it is always the case that

$$\|E\|_Y + \|\partial_q E\|_Y \leq \tau.$$

Let us also set  $\xi = \varepsilon q$ . We have now that

$$W_q(x) = w_\lambda(x - q), \quad \lambda = V(\xi).$$

We compute

$$J_\varepsilon(W_q) = J^\lambda(w_\lambda) + \frac{1}{2} \int_{\mathbb{R}^N} (V(\xi + \varepsilon x) - V(\xi)) w_\lambda^2(x) dx$$

where

$$J^\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} v (-\Delta)^s v + \frac{\lambda}{2} \int_{\mathbb{R}^N} v^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} v^{p+1}.$$

We recall that

$$w_\lambda(x) := \lambda^{\frac{1}{p-1}} w(\lambda^{\frac{1}{2s}} x)$$

satisfies the equation

$$(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0 \quad \text{in } \mathbb{R}^N.$$

where  $w = w_1$  is the unique radial least energy solution of

$$(-\Delta)^s w + w - w^p = 0 \quad \text{in } \mathbb{R}^N.$$

Then, after a change of variables we find

$$J^\lambda(w_\lambda) = \frac{1}{2} \int_{\mathbb{R}^N} w_\lambda (-\Delta)^s w_\lambda + \frac{\lambda}{2} \int_{\mathbb{R}^N} w_\lambda^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} w_\lambda^{p+1} = \lambda^{\frac{p+1}{p-1} - \frac{N}{2s}} J^1(w).$$

Now since  $w$  is radial, we find

$$\int_{\mathbb{R}^N} x_i w_\lambda(x) dx = 0.$$

Thus,

$$\int_{\mathbb{R}^N} (V(\xi + \varepsilon x) - V(\xi)) w_\lambda^2(x) dx = \nabla V(\xi) \cdot \int_{\mathbb{R}^N} x w_\lambda + O(\varepsilon^2) = O(\varepsilon^2)$$

On the other hand

$$\begin{aligned} & \partial_q \int_{\mathbb{R}^N} (V(\xi + \varepsilon x) - V(\xi)) w_\lambda^2(x) dx = \\ & \varepsilon \int_{\mathbb{R}^N} (\nabla V(\xi + \varepsilon x) - \nabla V(\xi)) w_\lambda^2(x) dx + \\ & 2 \int_{\mathbb{R}^N} (V(\xi + \varepsilon x) - V(\xi)) w_\lambda \partial_q w_\lambda dx = O(\varepsilon^2). \end{aligned}$$

**Lemma 6.3.** *Let  $\theta = \frac{p+1}{p-1} - \frac{N}{2s}$ ,  $c_* = J_1(w)$  and  $k = 1$ . Then the following expansions hold:*

$$\begin{aligned} I(q) &= c_* V^\theta(\xi) + O(\varepsilon^{\min(4s, 2)}) \\ \nabla_q I(q) &= c_* \varepsilon \nabla_\xi (V^\theta)(\xi) + O(\varepsilon^{\min(4s, 2)}). \end{aligned}$$

For the case  $k > 1$  and  $\min_{i \neq j} |q_i - q_j| \geq R \gg 1$ , we observe that, also,  $\|E\|_Y = O(\tau)$  and hence we also have

$$I(q) = J_\varepsilon(W_q) + O(\tau^2), \quad \partial_q I(q) = \partial_q J_\varepsilon(W_q) + O(\tau^2).$$

By expanding  $I(q)$  we get the validity of the following estimate.

**Lemma 6.4.** *Letting  $\xi = \varepsilon q$  we have that*

$$\begin{aligned} I(q) &= c_* \sum_{i=1}^k V^\theta(\xi_i) - \sum_{i \neq j} \frac{c_{ij}}{|q_i - q_j|^{N+2s}} + O(\varepsilon^{\min(4s, 2)} + \frac{1}{R^{2(N+2s-\mu)}}), \\ \nabla_q I(q) &= c_* \varepsilon \nabla_\xi \left[ \sum_{i=1}^k V^\theta(\xi_i) - \sum_{i \neq j} \frac{c_{ij}}{|q_i - q_j|^{N+2s}} \right] + O(\varepsilon^{\min(4s, 2)} + \frac{1}{R^{2(N+2s-\mu)}}) \end{aligned}$$

where  $c_*$  and  $c_{ij} = c_0(V(\xi_i))^\alpha (V(\xi_j))^\beta$  are positive constants.

*Proof.* It suffices to expand  $J_\varepsilon(W_q)$ . We see that, denoting  $w_i(x) := w_{\lambda_i}(x - q_i)$

$$\begin{aligned} J_\varepsilon(W_q) &= J_\varepsilon\left(\sum_{i=1}^k w_i\right) = \sum_{i=1}^k J_\varepsilon(w_i) + \\ & \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} w_i (-\Delta)^s w_j + \int_{\mathbb{R}^N} V(\varepsilon x) w_i w_j \\ & - \frac{1}{p+1} \int_{\mathbb{R}^N} \left(\sum_{i=1}^k w_i\right)^{p+1} - \sum_{i=1}^k w_i^{p+1}. \end{aligned} \quad (6.4)$$

We estimate for  $i \neq j$

$$\begin{aligned} & \int_{\mathbb{R}^N} w_i (-\Delta)^s w_j + \int_{\mathbb{R}^N} V(\varepsilon x) w_i w_j \\ &= \int_{\mathbb{R}^N} w_i w_j^p + \int_{\mathbb{R}^N} (V(\varepsilon x) - \lambda_j) w_i w_j \\ &= (c_{ij} + o(1)) \frac{1}{|q_i - q_j|^{N+2s}} + O\left(\frac{\varepsilon^{2s}}{R^{N+2s-\mu}}\right) \end{aligned} \quad (6.5)$$

where  $c_{ij} = c_0(V(\xi_i))^\alpha (V(\xi_j))^\beta$  and  $c_0, \alpha, \beta$  are constants depending on  $p, s$  and  $N$  only. Indeed,

$$w_i(x) = \lambda_i^{\frac{1}{p-1}} w(\lambda_i^{\frac{1}{2s}}(x - q_i))$$

and it is known that

$$w(x) = \frac{c_0}{|x|^{N+2s}}(1 + o(1)) \quad \text{as } |x| \rightarrow \infty.$$

Then, we have

$$\int_{\mathbb{R}^N} w_j^p w_i = \lambda_i^{\frac{1}{p-1} - \frac{n+2s}{2s}} \lambda_j^{\frac{p}{p-1} - \frac{n}{2s}} \left( \int_{\mathbb{R}^N} w^p \right) \frac{c_0}{|q_i - q_j|^{N+2s}},$$

and hence

$$c_{ij} = c_0 \lambda_i^\alpha \lambda_j^\beta$$

where

$$\lambda_i = V(\xi_i), \quad \lambda_j = V(\xi_j), \quad \alpha = \frac{1}{p-1} - \frac{n+2s}{2s}, \quad \beta = \frac{p}{p-1} - \frac{n}{2s}.$$

To estimate the last term we note that

$$\begin{aligned} \int_{\mathbb{R}^N} \left( \left( \sum_{i=1}^k w_i \right)^{p+1} - \sum_{i=1}^k w_i^{p+1} \right)^{p+1} &= \sum_{j=1}^k \int_{\Omega_j} \left( \left( \sum_{i=1}^k w_i \right)^{p+1} - \sum_{i=1}^k w_i^{p+1} \right)^{p+1} \\ &= \sum_{j=1}^k \sum_{\Omega_j} \left( (p+1) w_j^p \left( \sum_{i \neq j} w_i \right) + O(w_j^{\min(p-1,1)} \left( \sum_{i \neq j} w_i \right)^2) \right) \\ &= \sum_{j=1}^K \sum_{i \neq j} (p+1) \int_{\mathbb{R}^N} w_j^p w_i + O\left( \frac{1}{R^{2(N+2s-\mu)}} \right) \\ &= \sum_{j=1}^K \sum_{i \neq j} (p+1) \frac{c_{ij} + o(1)}{|q_i - q_j|^{N+2s}} + O\left( \frac{1}{R^{2(N+2s-\mu)}} \right) \quad (6.6) \end{aligned}$$

Substituting (6.5) and (6.6) into (6.4) and using the estimate of  $J_\varepsilon(w_i)$  in the proof of Lemma 6.3, we have estimated  $J_\varepsilon(w_i)$ , and we have proven the lemma.

## 7. THE PROOFS OF THEOREMS 1–3

Based on the asymptotic expansions in Lemma 6.4, we present the proofs of Theorems 1–3.

**Proof of Theorems 1 and 2.** Let us consider the situation in Remark 1.1, which is more general than that of Theorem 1. Then, in the definition of the configuration space  $\Gamma$  (3.3), we can take a fixed  $\delta$  and  $R \sim \varepsilon^{-1}$  and achieve that  $\Lambda \subset \varepsilon\Gamma$ . Then we get

$$\|E\|_Y + \|\partial_q E\|_Y = O(\varepsilon^{\min\{2s, 1\}}).$$

Letting

$$\tilde{I}(\xi) := I(\varepsilon q)$$

we need to find a critical point of  $\tilde{I}$  inside  $\Lambda$ . By Lemma 6.4, we see then that

$$\tilde{I}(\xi) - c_* \varphi(\xi) = o(1), \quad \nabla_\xi \tilde{I}(\xi) - c_* \nabla_\xi \varphi(\xi) = o(1),$$

uniformly in  $\xi \in \Lambda$  as  $\varepsilon \rightarrow 0$ , where  $\varphi$  is the functional in (1.10). It follows, by the assumption on  $\varphi$  that for all  $\varepsilon$  sufficiently small there exists a  $\xi^\varepsilon \in \Lambda$  such that  $\nabla \tilde{I}(\xi^\varepsilon) = 0$ , hence Lemma 6.1 applies and the desired result follows.

Theorem 2 follows in the same way. We just observe that because of the  $C^1$ -proximity, the same variational characterization of the numbers  $c$ , for the functional



$\tilde{I}(\xi)$  holds. This means that the critical value predicted in that form is indeed close to  $c$ . The proof is complete.  $\square$

**Proof of Theorem 3.** Finally we prove Theorem 3. Following the argument in [18], we choose the following configuration space

$$\Lambda = \{(\xi_1, \dots, \xi_k) / \xi_j \in \Gamma, \min_{i \neq j} |\xi_i - \xi_j| > \varepsilon^{1-\frac{s}{4}}\} \quad (7.1)$$

with  $\Gamma$  given by (3.3), and we prove the following Claim and then Theorem 3 follows from Lemma 6.1:

**Claim:** letting  $\xi = \varepsilon q$ , the problem

$$\max_{(\xi_1, \dots, \xi_k) \in \Lambda} I(q) \quad (7.2)$$

admits a maximizer  $(\xi_1^\varepsilon, \dots, \xi_k^\varepsilon) \in \Lambda$ .

We shall prove this by contradiction. First, by continuity of  $I(q)$ , there is a maximizer  $\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_k^\varepsilon) \in \bar{\Lambda}$ . We need to prove that  $\xi \in \Lambda$ . Let us suppose, by contradiction, that  $\xi^\varepsilon \notin \Lambda$ , hence it lies on its boundary. Thus there are two possibilities: either there is an index  $i$  such that  $\xi_i^\varepsilon \in \partial\Gamma$ , or there exist indices  $i \neq j$  such that

$$|\xi_i^\varepsilon - \xi_j^\varepsilon| = \min_{i \neq j} |\xi_i - \xi_j| = \varepsilon^{1-s}.$$

Denoting  $q^\varepsilon = \frac{\xi^\varepsilon}{\varepsilon}$ , and using Lemma 6.4, we have in the first case that

$$\begin{aligned} I(q^\varepsilon) &\leq c_* V^\theta(\xi_i^\varepsilon) + c_* \sum_{j \neq i} V^\theta(\xi_j^\varepsilon) + C\varepsilon^{2s} \leq \\ &c_* k \max_{\Gamma} V^\theta(x) + c_* (\max_{\partial\Gamma} V^\theta(x) - \max_{\Gamma} V^\theta(x)) + C\varepsilon^{2s}. \end{aligned} \quad (7.3)$$

In the second case, we invoke again Lemma 6.4 and obtain

$$I(q^\varepsilon) \leq c_* k \max_{\Gamma} V^\theta(x) - c_2 \varepsilon^{\frac{s}{4}} + C\varepsilon^{2s}$$

for some  $c_2 > 0$ . On the other hand, we can get an upper bound for  $I(q^\varepsilon)$  as follows. Let us choose a point  $\xi_0$  such that  $V(\xi_0) = \max_{\Gamma} V(x)$  and let

$$\xi_j = \xi_0 + \varepsilon^{1-\frac{1}{8}s}(1, 0, \dots, 0), \quad j = 1, \dots, k.$$

It is easy to see that  $(\xi_1, \dots, \xi_k) \in \Lambda$ . Now, we compute by Lemma 6.4:

$$I(q^\varepsilon) = \max_{\Lambda} I(q) \geq c_* k \max_{\Gamma} V^\theta(x) - c_3 \varepsilon^{\frac{s}{8}}. \quad (7.4)$$

For  $\varepsilon$  sufficiently small, a contradiction follows immediately from (7.3)-(7.4).  $\square$

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- After completion of this work we have learned about the paper [4] in which the result of Corollary 1.1 is found for  $k = 1$  under further constraints in the space dimension  $N$  and the values of  $s$  and  $p$ .

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DEPARTAMENTO DE INGENIERÍA MATEMÁTICA AND CENTRO DE MODELAMIENTO MATEMÁTICO  
(UMI 2807 CNRS), UNIVERSIDAD DE CHILE, CASILLA 170 CORREO 3, SANTIAGO, CHILE.

*E-mail address:* `jdavila@dim.uchile.cl`

DEPARTAMENTO DE INGENIERÍA MATEMÁTICA AND CENTRO DE MODELAMIENTO MATEMÁTICO  
(UMI 2807 CNRS), UNIVERSIDAD DE CHILE, CASILLA 170 CORREO 3, SANTIAGO, CHILE.

*E-mail address:* `delpino@dim.uchile.cl`

DEPARTMENT OF MATHEMATICS, CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG  
AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B.C.,  
CANADA, V6T 1Z2.

*E-mail address:* `wei@math.cuhk.edu.hk`